Correspondences

Recall a correspondence between X and Y is a cycle in $X \times Y$. Namely if $\dim X = d$, and $A \in Z^r(X \times Y)$ is a correspondence, this gives something like a map:

$$Z^t(X) \ni T \longmapsto A(T) = (p_Y)_*(A \cdot (T \times Y)) \in Z^{d+t-r}(Y)$$

If we pass to classes via an adequate equivalence relation, then A actually induces group homomorphisms on the resulting groups.

If given such a correspondence $A \in CH(X \times Y, \mathbb{Q})$ (note now we pass to classes), we can define a transpose ${}^TA$ as the image of A under the obvious morphism $CH(X \times Y, \mathbb{Q}) \xrightarrow{\sim} CH(Y \times X, \mathbb{Q})$. We now also set $Corr(X,Y) = CH(X \times Y, \mathbb{Q})$.

Given $f \in Corr(X,Y)$, $g \in Corr(Y,Z)$, we set $g \circ f = (P_{X \times Z})_*((f \times Z) \cdot (X \times g))$. Checking that this is an associative composition law is tedious, one could use "Lieberman's Lemma" which is also tedious & technical, but not difficult.

Lemma: If $f \in Corr(X,Y)$, $\alpha \in Corr(X,X')$, and $\beta \in Corr(Y,Y')$, then we have $(\alpha \times \beta)_* f = \beta \circ f \circ {}^T\alpha$.

We leave it to the reader to fill in the gaps (and find the identity correspondence). We call this new category $C_\sim SmProj_k$, and is already additive! Any correspondence of degree r induces morphisms $f_* : C^i_\sim(X, \mathbb{Q}) \longrightarrow C^{i+r}_\sim(Y, \mathbb{Q})$, $Z \mapsto (pr_Y)_*(f \cdot (pr_X)^*(Z))$. If our equivalence relation is finer than homological, we get $2r$-degree maps on the chosen Weil cohomology. (Note the ramifications of conjecture D!).

Def: A projector is an element $p \in Corr(X,X)$ s.t. $p \circ p = p$.

From now on, X is $\in SmProj_k$, irred., $\dim X = d$. Then $Corr^t(X,Y) = C^{d+t}_\sim(X \times Y, \mathbb{Q})$. Note all projectors are contained in $Corr^0(X,X)$.

Motives

Def: The category Mot_k^{eff} is the pseudo-abelian completion, i.e. objects are pairs (X, p), $X \in SmProj_k$, and p a projector, and $Hom((X,p),(Y,q)) = q \circ \{Corr^0(X,Y)\} \circ p$. The category Mot_k consist of triples (X, p, n), where $n \in \mathbb{Z}$ and everything else the same. Morphisms are $Hom((X,p,m),(Y,q,n)) = q \circ Corr^{n-m}(X,Y) \circ p$.

If $\sim = \sim_{rat}$, then this is the category of Chow motives. If $\sim = \sim_{num}$, then its called numerical motives. In general, we get a functor:

$$\left.\begin{array}{l} h_\sim : SmProj_k \longrightarrow Mot_k^\sim \\ \qquad X \longmapsto (X, \Delta_X, 0) \\ (f: X \to Y) \mapsto {}^T\Gamma_f : h_\sim(Y) \to h_\sim(X). \end{array}\right\} \sim \text{ is an adequate equivalence relation.}$$

In the special case of $\sim = \sim_{rat}$, we denote $h_\sim(-)$ as $ch(-)$. Lets now look at some examples of motives.

Examples

1) The easiest example is of course the motive of a point: $h_\sim(*) = (\mathrm{Spec}\, k, \mathrm{id}, 0)$. We typically denote this as $\mathbb{1}$.

2) Let X be irreducible of dimension d, and assume $X_d(k) \neq \emptyset$. Choose $e \in X(k)$, and define $p_0(X) = e \times X$ and $p_{2d}(X) = X \times e$. We claim that these are projectors. Indeed, denoting by $\pi_{13} : X \times X \times X \to X \times X$ the projection omitting the middle factor:

$$
\begin{aligned}
p_0(X) \circ p_0(X) &= (\pi_{13})_* \left((p_0(X) \times X) \cdot (X \times p_0(X)) \right) \\
&= (\pi_{13})_* \left((e \times X \times X) \cdot (X \times e \times X) \right) \\
&= (\pi_{13})_* (e \times e \times X)
\end{aligned}
$$

Now since $\dim(\pi_{13}(e \times e \times X)) = \dim(e \times X)$, and $\deg(e \times e \times X / \pi_{13}(e \times e \times X)) = 1$, this equals $p_0(X)$. The other is similar. Moreover since $\dim(e \times X \times e) > \dim(e \times e)$, we also see that $p_0(X) \circ p_{2d}(X) = p_{2d}(X) \circ p_0(X) = 0$, so these are "orthogonal projectors". So we define

$$h^0_\sim(X) = (X, p_0(X), 0) \quad \text{and} \quad h^{2d}_\sim (X, p_{2d}(X), 0).$$

In $\mathrm{Mot}_k^\sim$, we know already that $\mathrm{Hom}((X,p,0),(X',q,0)) = q \circ \mathrm{Corr}^0(X,Y) \circ p$, and in particular if $(X,p,0) = (X',q,0)$, then $\mathrm{id} \in \mathrm{Hom}(\text{—})$ is given by just p. Hence Two motives M and M' (both with integer zero) are isomorphic if we have maps f, g $M \underset{g}{\overset{f}{\rightleftarrows}} M'$ with $g \circ f = q = \mathrm{id}_{M'}$ & $f \circ g = \mathrm{id}_M$. This means we have two correspondences f', g' s.t.

$$p \circ g' \circ q \circ f' \circ p = p \quad \text{and} \quad q \circ f' \circ p \circ g' \circ q = q.$$

Now we claim that regardless of X (in SmProj_k, irreducible, dimension d), $h^0_\sim(X) \cong \mathbb{1}$. Indeed consider $e : \mathrm{Spec}\, k \to X$ and the structure map $s : X \to \mathrm{Spec}\, k$. Then compute $\mathrm{id} \circ e^* \circ p_0 \circ s^* \circ \mathrm{id}$. You can check by hand that this reduces to $e^* \circ s^*$, which is:

$$
\begin{aligned}
e^* \circ s^* &= (\pi_{k \times k})_* \left(({}^T\Gamma_s \times \mathrm{Spec}\, k) \cdot (\mathrm{Spec}\, k \times {}^T\Gamma_e) \right) \\
&= (\pi)_* \left((\mathrm{Spec}\, k \times X \times \mathrm{Spec}\, k) \cdot (\mathrm{Spec}\, k \times e \times \mathrm{Spec}\, k) \right) \\
&= (\pi)_* (\mathrm{Spec}\, k \times e \times \mathrm{Spec}\, k) = \mathrm{Spec}\, k \times \mathrm{Spec}\, k = \mathrm{id} \ (\text{projector on } \mathbb{1}).
\end{aligned}
$$

Checking the other composition is an exercise. Also of note: $h^{2d}_\sim(X_d) \cong h^{2d}_\sim(Y_d)$ for all pairs.

3) If $M = (X, p, m)$, $N = (Y, q, m)$, then we can also define $M \oplus N = (X \sqcup Y, p+q, m)$. The definition w/ different integers is skipped. However we can check that for X (again, irred.), setting $p^+(X) = \Delta - p_0(X) - p_{2d}(X)$, we get: $h_\sim(X) = h^0_\sim(X) \oplus h^+_\sim(X) \oplus h^{2d}_\sim(X)$.

4) Two important motives are the Lefschetz and Tate motives, defined via:

$$\mathbb{L}_\sim = (\mathbb{P}^1, \mathbb{P}^1 \times e, 0) = h^2_\sim(\mathbb{P}^1) \quad \text{and} \quad \mathbb{T}_\sim = (\mathrm{Spec}\, k, \mathrm{id}, 1).$$

It actually turns out that $\mathbb{L}_\sim \cong (\mathrm{Spec}\, k, \mathrm{id}, -1)$ in addition. Here (I am told) that $\mathbb{L}_\sim$ plays the role of a "fundamental class" and $\mathbb{T}_\sim$ gives tate twists.

5) If $M = (X, p, 0)$, then $1-p$ is also a projector, and $h_\sim(X) = (X,p,0) \oplus (X, 1-p, 0)$.

Some more remarks are in order: Namely Mot_k is a $\mathbb{Q}$-linear psuedo-abelian category (if we take $\mathbb{Q}$-coefficients), and is a tensor category (or at least, has a tensor operation) given by:

$$(X, p, n) \otimes (Y, q, m) = (X \times Y, p \times q, n+m).$$

If we use the unproven fact that $\mathbb{L}_\sim \cong (\mathrm{Spec}\, k, id, -1)$, we see that $\mathbb{L}_\sim \otimes \mathbb{T}_\sim = \mathbb{1}$. Further $\mathbb{L}_\sim^{\otimes d} \cong h^{2d}(X_d)$ for all X irred. of dim. d (hence the fundamental class remark earlier). We also see that $(X, p, m) \cong (X, p, 0) \otimes \mathbb{T}_\sim^{\otimes m}$.

On Mot_k, there is also a duality operator $D: Mot_k^{op} \to Mot_k$, sending (X_d, p, m) to $(X_d, {}^T p, d-m)$. This is an involution, and allows us to define

$$h_*^\sim = D \circ h_\sim,$$

which is a covariant functor. This is important for a homological approach, rather then cohomological. One other remark: there are non isomorphic varieties with isomorphic motives.

Cohomology of Motives

For any projector $p: X \to X$, we have induced maps $p_*: CH^i(X)_\mathbb{Q} \longrightarrow CH^i(X)_\mathbb{Q}$. Hence we can define the i^{th} Chow group of a motive $M = (X, p, m)$ as

$$CH^i(M) = \mathrm{Im}\, p_* \subset CH^{i+m}(X)_\mathbb{Q}$$

We could also have defined $CH^i(M) = \mathrm{Hom}_{Mot_k}(\mathbb{L}^{\otimes i}, M)$, indeed:

$$\begin{aligned} \mathrm{Hom}_{Mot_k}(\mathbb{L}^{\otimes i}, M) &= \mathrm{Hom}((\mathrm{Spec}\, k, id, -i), (X, p, m)) = p \circ \mathrm{Corr}^{m+i}(\mathrm{Spec}\, k, X) \\ &= \{ p \circ \Gamma \mid \Gamma \in CH^{i+m}(X) \} \end{aligned}$$

Now by Lieberman's lemma, $p \circ \Gamma = (id_{\mathrm{Spec}\, k} \times p)_* \Gamma$, so this is the same as $\mathrm{Im}\, p_*$. There was nothing special about $\sim_{rat}$ above. Can define all cycle groups using $\mathrm{Hom}(\mathbb{L}^{\otimes i}, M)$.

If we have $\sim$ finer or equal to $\sim_{hom}$, then projectors act on a Weil cohomology theory via $\alpha \longmapsto p_*(\alpha) = p_X\{ cl_{X \times X}(p) \cup p_X^*(\alpha) \}$. Hence for $M = (X, p, m)$, we can define

$$H^i(M) = \mathrm{Im}\, p_* \subset H^{i+2m}(X).$$

Note that unless $D(-)$ is true, we cannot give Mot_k^{Num} cohomology groups. The subject of motivic cohomology is far from trivial, so we won't discuss it here.

Motives of Curves

Here we work specifically with coefficients in $\mathbb{Z}$, and an algebraically closed field of an appropriate characteristic.

Let C be a curve over k, assumed smooth and projective. If we set $p_1(C) = p^+(C) = \Delta_C - p_0(C) - p_2(C)$, then a so-called Chow-Künneth decomposition:

$$ch(C) = Ch^0(C) \oplus ch^1(C) \oplus ch^2(C).$$

We have already seen (or at least have seen claimed) that $ch^0(C) \cong \mathbb{1}$ and $ch^2(C) \cong ch^2(C')$ for any other smooth projective curve, then $ch^1(C)$ had better be interesting for the theory to mean anything. As it turns out, it's extremely interesting.

Theorem:
a) The only nontrivial chow group of $ch^1(C)$ is $CH^1_{alg}(C) = J(C)(k)$,
b) Let C, C' be smooth projective curves, then:

$$\mathrm{Hom}_{Mot}(ch^1(C), ch^1(C')) = \mathrm{Hom}_{AbVar}(J(C), J(C')), \quad \text{and}$$

c) In $Mot_{rat} = \mathrm{Chow\,Mot}_{\mathbb{Q}}$, let $\mathcal{M}$ be the full subcategory whose objects are direct sums of $ch^1(C)$, C some smooth projective curve. Then $\mathcal{M}$ is equivalent to the category of abelian varieties up to rational isogeny.

Sketch of Proof:
(a) It is near obvious the only nontrivial one is $CH^1_{alg}(C)$. Given $\alpha \in CH^1(C)$, we see $p_{1*}(\alpha) = \alpha - \deg(\alpha)e$, e a k-rational point. Thus $\deg(p_{1*}(\alpha)) = 0$, which is in $J(C)(k)$.

(b) Now if X is in $SmProj_k$, and $x_0 \in X(k)$, then Alb_X is the universal abelian variety w/ respect to maps $X \to Ab.Var.$, taking $x_0 \to 0 \in Alb_X$. Now Pic_X is the picard scheme, and Pic^0_X the connected component of the identity. Then there is a correspondence P_X in $CH^1(P_X \times X)$, taking $P_X(0) = 0$, ${}^T P_X(x_0) = 0$, which is final among pairs (S, D) where $D \in CH^1(S \times X)$, ${}^T D(x_0) = 0$ and $D(s_0) \in CH^1_{alg}(X)$, $s_0 \in S(k)$.

Now if (Y, y) is another based k-variety, then:

$$\mathrm{Hom}^0_{SmProj}(X, Pic^0_Y) = \mathrm{Hom}_{AV}(Alb_X, Pic^0_Y) \cong A := \{ D \in CH^1(X \times Y) \mid D(x_0) = 0,\ {}^T D(y_0) = 0 \}.$$

Note $A \hookrightarrow CH^1(X \times Y)$ is not in general an isomorphism, the kernel being divisors of the product which do not project onto all of X or Y. All such can be written as $D = D_1 \times Y + X \times D_2$.

For smooth curves, Abel's theorem gives $Alb_C \cong J(C) \cong Pic^0_C$. Now we prove (b). We have:

$$\psi_1 : CH^1(C \times C') \longrightarrow \mathrm{Hom}_{Mot}(ch^1(C), ch^1(C'))$$
$$T \longmapsto p_1(C') \circ T \circ p_1(C).$$

The kernel is exactly degenerate divisors. Hence by the above, equal to $\mathrm{Hom}_{AV}(Pic^0_C, Pic^0_{C'})$

Now for (c). We claim the following: If A is an Ab. Var., then there exists a curve C and a map $J(C) \twoheadrightarrow A$. We also claim: Let A be an Ab. Var. and a sub. Ab. Var. B. Then $\exists$ $C \subseteq A$ w/ $B \cap C$ finite and $C \oplus B \xrightarrow{\sim} A$.

So given $J(C) \twoheadrightarrow A$, taking duals gives some $A_1 \leq J(C)$ isogenous to $J(C)$, and an A_2 s.t. $A_1 \oplus A_2 \xrightarrow{\sim} J(C)$. Now general nonsense about the Karoubian completion gives the claim, under the functor $ch'(C) \longrightarrow J(C)$ and part (b) showed fully faithful w/ rational coefficients. ∎